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# New Internal Model Average Consensus Estimators with Light Communication Load

**Juwon Lee and Juhoon Back***

**Abstract:** The dynamic average consensus problem for a group of agents is considered. Each agent in the group is supposed to estimate the average of inputs applied to all agents and the estimation should be done in a distributed way. By reinterpreting the proportional integral type estimator, a new structure for the average estimator which can embed the internal model of inputs is proposed and conditions which result in the zero estimation error in the steady state are derived. We present constructive design procedures for the cases of constant inputs and time-varying inputs employing the root locus for the former and LQR-based design for the latter. The theory is validated through numerical simulations.

**Keywords:** Consensus, distributed system, multi-agent, network system, state estimation.

## 1. INTRODUCTION

Thanks to remarkable achievements in various fields of engineering and science including communication, computation, electronics, it is now possible to see in our daily life that multiple agents, often called multi-agent systems, exchange information in real time, make decisions autonomously, and act cooperatively. One of fundamental issues related to multi-agent systems is the consensus problem in which agents try to reach an agreement by exchanging information through communication network, and a vast amount of results can be found in the literature, see the early research articles [1–3], books [4–6], and surveys [7, 8].

In this paper, we consider the dynamic average consensus problem for a group of agents. Suppose that each agent in the group is endowed with an input and we would like to estimate the average of inputs applied to all the agents. The inputs can be constants or time-varying, and the agents are allowed to communicate with neighboring agents so that the estimation is done in a distributed way. It is noted that the problem is somewhat different from the consensus or synchronization problem [7] where the objective is to synchronize the states or outputs of agents and the synchronized trajectory depends on the initial conditions of agents (and sometimes those of controllers as well), while in dynamic average consensus problem the agreement should be made on the estimate of the average of inputs so that the estimate should be independent of the initial state of dynamic system (filters or controllers) running on agents. Applications of the problem include sensor fusion [9, 10], distributed estimation [11, 12], economic dispatch [13, 14], resource allocation [15], etc.

A number of results on dynamic average consensus problems are available in the literature. In [16], the authors presented a solution for the case of constant inputs exploiting the fact that sum of all states remains constant in a group of single integrators with diffusive coupling (and under undirected network topology), and proposed a solution for the case where the inputs are polynomial-in-time signals. The internal model based estimator is presented in [17] and nonlinear protocols under time-varying network topology are proposed in [18]. Discrete-time estimation algorithms are presented in [19, 20] and time-varying network as well as privacy of agents is considered in [21]. Recently, a tutorial considering applications has been published [22].

The most relevant work to ours is the internal model based estimator [17] which is a generalization of proportional integral (PI) type estimator given in [23]. Although the internal model of inputs [24], e.g., the order of polynomial or the frequency of sinusoids, should be known completely, it is the main advantage that the estimation error asymptotically converges to zero. Moreover, the result is essentially global in the sense that the bounds of inputs and/or their time derivatives are not required, and the initial condition of estimator can be freely chosen. Despite these advantages, it is the main drawback that estimator

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Juwon Lee and Juhoon Back are with the School of Robotics, Kwangwoon University, 20 Kwangwoon-ro, Nowon-gu, Seoul 01897, Korea (e-mails: show8765@kw.ac.kr, backhoon@kw.ac.kr).
* Corresponding author.

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design is not fully constructive since the simultaneous stabilization problem (the problem of finding one stabilizing controller for several systems) involved in the design was not solved completely in [17].

In this paper, we reinterpret the PI-type estimator of [23] and propose a new structure for dynamic average estimator. Compared to [17, 23], the proposed structure is simpler and asks for less communication among agents, which is highly desirable in practice. In addition, the simpler structure enables us to develop a constructive design procedure for the estimator. Based on the new estimator structure, two solutions are presented. One is for the constant input case, and it is designed employing the root locus under the assumption that an upper bound of the largest eigenvalue of the Laplacian is known. The other is for the time-varying input case, which adopts the LQR design assuming that a lower bound of the smallest nonzero eigenvalue of the Laplacian is known. A rigorous stability analysis for the closed-loop system is given and numerical simulations are conducted to validate the theory.

The remainder of this paper is organized as follows: In Section 2, a new structure for dynamic average estimator is proposed and a solution to constant input case is presented. In Section 3, we consider the case with time-varying inputs and develop a constructive design procedure. After presenting numerical simulation results in Section 4, we give some conclusion in Section 5.

**Notation:** Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{C}$ denote the set of complex numbers. $\mathbb{C}_{<0}$ is the set of complex numbers with negative real part. $0_k$ stands for the zero vector in $\mathbb{R}^k$, $1_k \in \mathbb{R}^k$ a vector with all components being 1. $I_k$ represents the identity matrix in $\mathbb{R}^{k\times k}$. Concatenation of two vectors or scalars $a$ and $b$ is denoted by $[a;b]$, i.e., $[a;b] := [a^\top, b^\top]^\top$. Given $n$ scalars $a_1,\dots,a_n$, $\mathrm{diag}\{a_1,\dots,a_n\}$ denotes the diagonal matrix whose diagonal elements are $a_1,\dots,a_n$, while all the other elements are zero. The block diagonal matrix for $n$ matrices $A_1,\dots,A_n$, denoted by $\mathrm{diag}\{A_1,\dots,A_n\}$, is similarly defined. Given a function $x(t)$, $x(s)$ denotes the Laplace transform of $x(t)$. We also express $x(s) = \mathcal{L}\{x(t)\}$. Laplace transform of a vector valued function is similarly defined. For a symmetric matrix $M \in \mathbb{R}^{n\times n}$, $\lambda_i(M)$, $i = 1,\dots,n$, denote the eigenvalues of $M$ arranged as $\lambda_1(M) \le \cdots \le \lambda_n(M)$.

## 2. NEW AVERAGE ESTIMATOR FOR CONSTANT INPUTS

Consider a team of $N$ agents and suppose that the $i$th agent in the team, $i = 1,\dots,N$, can access or measure a signal $u^i(t) \in \mathbb{R}$. In this section, we assume that $u^i(t)$ is constant, $u^i(t) = \bar{u}^i, \forall t$, with $\bar{u}^i \in \mathbb{R}$.

The problem under consideration is to compute or estimate the average of $u^i$'s, defined by $u_{\mathsf{av}} = \frac{1}{N}\sum_{i=1}^N u^i$, in a distributed manner. That is to say, the average is not computed in a centralized way, i.e., a unit collects all $u^i$'s and compute the average. Instead, each agent runs an estimator (or a filter) which produces an estimate of $u_{\mathsf{av}}$ and this is done by exchanging filtered signals of $u^i$'s, denoted by $v^i$'s, with its neighbors. In what follows, the estimate of $u_{\mathsf{av}}$ by agent $i$ is denoted by $\hat{u}^i_{\mathsf{av}}$.

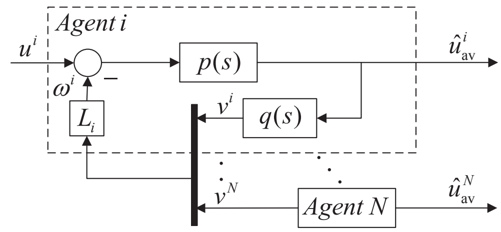

Fig. 1. Proposed structure for the agent $i$.

The communication among agents is modeled by a graph $\mathcal{G}$ and it is assumed that the graph is undirected and connected. Let $A = [a_{ij}] \in \mathbb{R}^{N\times N}$ be the adjacency matrix associated to $\mathcal{G}$ where $a_{ij}$ represents the weight of interconnection and defined by $a_{ij} = a_{ji} > 0$ if agents $i$ and $j$ can communicate each other, $a_{ij} = a_{ji} = 0$ if they can't, and $a_{ii} = 0$, $i = 1,\dots,N$. The Laplacian of the graph $\mathcal{G}$, denoted by $L$, is an $N\times N$ matrix whose components are defined by $l_{ij} = -a_{ij}$ if $i \ne j$ and $l_{ii} = \sum_{j=1}^N a_{ij}$. $\mathcal{N}_i$ indicates the set of neighbors of the $i$-th agent.

Now we present a distributed estimator which can asymptotically compute $u_{\mathsf{av}}$. Fig. 1 describes the proposed estimator for each agent. The estimator of agent $i$ is a feedback system composed of two filters $p(s)$ and $q(s)$, $p(s)$ in the feedforward path and $q(s)$ in the feedback path. It takes the signal $u^i$ as input and produces $\hat{u}^i_{\mathsf{av}}$ which is an estimate of $u_{\mathsf{av}}$. The signals $v^i$'s, which are filtered signals of $\hat{u}^i_{\mathsf{av}}$'s, are exchanged among agents so that the signal $\omega^i = L_i v$, where $v(t) = [v^1(t);\cdots;v^N(t)]$ and $L_i$ is the $i$th row vector of the Laplacian $L$, is fed back to the agent $i$'s estimator. It is compared with $u^i$ and the difference $u^i - \omega^i$ is passed to $p(s)$.

In what follows, we use vectors $u(t)$, $\hat{u}_{\mathsf{av}}(t)$, and $\omega(t)$, which collect information from $N$ agents, e.g., $u(t) = [u^1(t);\cdots;u^N(t)]$. The vectors $\hat{u}_{\mathsf{av}}(t)$ and $\omega(t)$ are defined similarly.

The filters $p(s)$ and $q(s)$ are given by

$$p(s) = \frac{b_{\mathsf{p},n-1}s^{n-1} + \cdots + b_{\mathsf{p},0}}{s^n + a_{\mathsf{p},n-1}s^{n-1} + \cdots + a_{\mathsf{p},0}},$$
$$q(s) = \frac{b_{\mathsf{q},m-1}s^{m-1} + \cdots + b_{\mathsf{q},0}}{s^m + a_{\mathsf{q},m-1}s^{m-1} + \cdots + a_{\mathsf{q},0}}, \quad (1)$$

where $n$, $m$ are positive integers and the coefficients of $p(s)$, $q(s)$ are design parameters.

Assume that the numerator and denominator of $p(s)$ are coprime and assume the same for $q(s)$. Throughout the paper, we assume coprimeness of the numerator and denominator of a transfer function unless stated otherwise. In the state space, the proposed estimator can be realized

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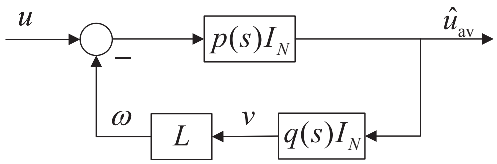


Fig. 2. Proposed structure for dynamic average estimator for all agent. The Laplacian $L$ explains how the signals $v^1,\dots,v^N$, generated by filters, are exchanged among agents.

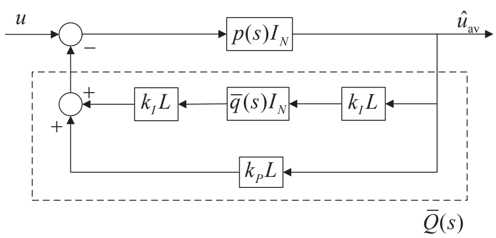


Fig. 3. Dynamic average estimator proposed in [17, 23]. Two blocks $k_I L$ indicates that both $\hat{u}_{\mathsf{av}}$ and the output of $\bar{q}(s)$ should be exchanged through communication.

as

$$\begin{aligned} \dot{\xi}^i &= A_\xi \xi^i + B_\xi \left(u^i - \omega^i\right), \quad \hat{u}^i_{\mathsf{av}} = C_\xi \xi^i, \\ \dot{\eta}^i &= A_\eta \eta^i + B_\eta \hat{u}^i_{\mathsf{av}}, \quad v^i = C_\eta \eta^i, \\ \omega^i &= \sum_{j\in\mathcal{N}_i} a_{ij}(v^i - v^j) \end{aligned} \tag{2}$$

where $\xi^i \in \mathbb{R}^n$, $\eta^i \in \mathbb{R}^m$, and the matrices are defined by

$$\begin{aligned} A_\xi &= \left[\begin{array}{c|c} 0_{n-1} & I_{n-1} \\ \hline -a_{\mathsf{p},0} & -a_{\mathsf{p},1} \cdots -a_{\mathsf{p},n-1} \end{array}\right], \quad B_\xi = \begin{bmatrix} 0_{n-1} \\ 1 \end{bmatrix}, \\ C_\xi &= \begin{bmatrix} b_{\mathsf{p},0} & b_{\mathsf{p},1} & \cdots & b_{\mathsf{p},n-1} \end{bmatrix}, \\ A_\eta &= \left[\begin{array}{c|c} 0_{m-1} & I_{m-1} \\ \hline -a_{\mathsf{q},0} & -a_{\mathsf{q},1} \cdots -a_{\mathsf{q},m-1} \end{array}\right], \quad B_\eta = \begin{bmatrix} 0_{m-1} \\ 1 \end{bmatrix}, \\ C_\eta &= \begin{bmatrix} b_{\mathsf{q},0} & b_{\mathsf{q},1} & \cdots & b_{\mathsf{q},m-1} \end{bmatrix}. \end{aligned}$$

The proposed structure is a simpler implementation of the internal model based estimator introduced in [17, 23]. To obtain this, we first interpret the estimator as a feedback system composed of $p(s)$ and $\bar{Q}(s) = k_P L + k_I^2 L^2 \bar{q}(s)$ as shown in Fig. 3. ($p(s)$ and $\bar{q}(s)$ correspond to $h(s)$ and $g(s)$ in [17], respectively.) Then, it is observed that the convergence proof of [17] essentially requires three properties 1) $p(0) = 1$, 2) the transfer function $\bar{q}(s)$ has at least one pole at the origin, and 3) the closed loop system is stable. From this observation, we replace $\bar{Q}(s)$ in Fig. 3 by $Lq(s)$ to have a simpler structure shown in Fig. 2.

It is emphasized that the communication load is reduced in the proposed structure since only the outputs of $q(s)$ are exchanged, while not only the outputs of $p(s)$ but also those of $\bar{q}(s)$ should be exchanged in the previous one. In addition, as can be seen shortly, the proposed structure enables us to develop a constructive design procedure which has not been provided in [17, 23].

**Remark 1:** The communication load of the proposed estimator is lightest among existing average estimators including [16, 18, 21] because agents exchange only the outputs of the filter $q(s)$. In addition, the proposed estimator is robust in the sense that the initial conditions of the filters can be set arbitrarily, while those of [16, 18] require fixed initial conditions.

Now we derive the condition under which the average can be estimated asymptotically. Towards this, we first rewrite the dynamics of estimator (2) as

$$\begin{aligned} \hat{u}^i_{\mathsf{av}}(s) &= p(s)(u^i(s) - \omega^i(s)) = p(s)(u^i(s) - L_i v(s)), \\ v^i(s) &= q(s)\hat{u}^i_{\mathsf{av}}(s), \end{aligned} \tag{3}$$

where $L_i$ is the $i$th row of $L$, and the relation $\omega^i(s) = \sum_{j\in\mathcal{N}_i} a_{ij}(v^i(s) - v^j(s)) = L_i v(s)$ is used. We then rewrite the $N$ equations of (3) compactly as

$$\begin{aligned} \hat{u}_{\mathsf{av}}(s) &= p(s)(u(s) - Lv(s)), \\ v(s) &= q(s)\hat{u}_{\mathsf{av}}(s), \end{aligned}$$

from which we have

$$\hat{u}_{\mathsf{av}}(s) = \left(I_N + Lp(s)q(s)\right)^{-1} p(s)u(s) =: T(s)u(s). \tag{4}$$

**Theorem 1:** Let $\bar{u} \in \mathbb{R}^N$ be a constant vector and $u(t) = \bar{u}, \forall t \geq 0$. Suppose the estimator (2) is designed such that conditions C1, C2, and C3 shown below are satisfied.

C1) $p(s)$ is stable and $p(0) = 1$.

C2) $q(s)$ contains at least one pole at the origin, and all the other poles belong to $\mathbb{C}_{<0}$.

C3) All the roots of $1 + p(s)q(s)\lambda_i(L) = 0,\ i = 2, \cdots, N$, belong to $\mathbb{C}_{<0}$.

Then, it holds that

$$\lim_{t\to\infty} \hat{u}_{\mathsf{av}}(t) = \frac{1}{N} 1_N 1_N^\top \bar{u}.$$

**Proof:** Let $\lambda_1, \dots, \lambda_N$ be the eigenvalues of $L$, and $V$ be the orthonormal matrix such that $V^\top L V = \Delta$, where $\Delta = \mathrm{diag}\{0, \lambda_2, \dots, \lambda_N\}$. Then, from (4), one has

$$T(s) = VV^\top T(s) VV^\top = V \begin{bmatrix} p(s) & 0_{N-1}^\top \\ 0_{N-1} & \bar{T}(s) \end{bmatrix} V^\top,$$

where

$$\bar{T}(s) = \mathrm{diag}\left\{ \frac{p(s)}{1 + p(s)q(s)\lambda_2}, \dots, \frac{p(s)}{1 + p(s)q(s)\lambda_N} \right\}.$$

From conditions C1 and C3, it follows that $T(s)$ is stable, and thus the signal $\hat{u}_{\mathsf{av}}(t)$ has a steady state value

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$\hat{u}_{\mathsf{av},ss} = \lim_{t\to\infty} \hat{u}_{\mathsf{av}}(t)$. Note that since $T(s)$ is stable, $\hat{u}_{\mathsf{av},ss}$ is independent of the initial conditions of $\xi^i$'s and $\eta^i$'s. Recalling that the Laplacian $L$ has a zero eigenvalue and the associated eigenvector is of the form $c1_N$ with $c \neq 0$, we decompose $V$ as $V = \begin{bmatrix} \frac{1}{\sqrt{N}} 1_N & W \end{bmatrix}$. Applying the final value theorem, one has

$$\begin{aligned}
\hat{u}_{\mathsf{av},ss} &= \lim_{s\to 0} sT(s)\frac{\bar{u}}{s} \\
&= T(0)\bar{u} \\
&= \begin{bmatrix} \frac{1}{\sqrt{N}} 1_N & W \end{bmatrix} \begin{bmatrix} p(0) & 0_{N-1}^\top \\ 0_{N-1} & \bar{T}(0) \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{N}} 1_N^\top \\ W^\top \end{bmatrix} \bar{u} \\
&= \frac{1}{N} 1_N 1_N^\top p(0)\bar{u} + W\bar{T}(0)W^\top \bar{u}.
\end{aligned}$$

From conditions C1 and C2, it follows that $\frac{p(s)}{1+p(s)q(s)\lambda_i}$, $i = 2,\ldots,N$, has at least one zero at the origin, so that

$$\lim_{s\to 0} \frac{p(s)}{1+p(s)q(s)\lambda_i} = 0, \quad i = 2,\ldots,N,$$

which results in that $W^\top \bar{T}(0)W\bar{u} = 0$. Finally, the condition C1 ($p(0) = 1$) ensures that $\hat{u}_{\mathsf{av},ss} = \frac{1}{N} 1_N 1_N^\top \bar{u}$, which completes the proof. □

The novelty of the proposed structure lies in the fact that one can systematically find $p(s)$ and $q(s)$ such that the conditions in Theorem 1 are satisfied. A procedure to design the distributed estimator is given as follows.

**Design Procedure for Constant Inputs**

**Step 1:** Choose a positive integer $n$. Take stable polynomials $a_{\mathsf{p}}(s) = s^n + a_{\mathsf{p},n-1}s^{n-1} + \cdots + a_{\mathsf{p},0}$ and $\bar{b}_{\mathsf{p}}(s) = s^{n-1} + \bar{b}_{\mathsf{p},n-2}s^{n-2} + \cdots + \bar{b}_{\mathsf{p},0}$. Take $b_{\mathsf{p},n-1}$ such that $b_{\mathsf{p},n-1}\bar{b}_{\mathsf{p},0} = a_{\mathsf{p},0}$. Let $p(s) = \frac{b_{\mathsf{p}}(s)}{a_{\mathsf{p}}(s)} = \frac{b_{\mathsf{p},n-1}\bar{b}_{\mathsf{p}}(s)}{a_{\mathsf{p}}(s)}$.

**Step 2:** Choose a positive integer $m$. Take stable polynomials $\bar{a}_{\mathsf{q}}(s) = s^{m-1} + \bar{a}_{\mathsf{q},m-2}s^{m-2} + \cdots + \bar{a}_{\mathsf{q},0}$ and $\bar{b}_{\mathsf{q}}(s) = s^{m-1} + \bar{b}_{\mathsf{q},m-2}s^{m-2} + \cdots + \bar{b}_{\mathsf{q},0}$. Let $q(s) = \frac{b_{\mathsf{q},m-1}\bar{b}_{\mathsf{q}}(s)}{s\bar{a}_{\mathsf{q}}(s)}$ where $b_{\mathsf{q},m-1} > 0$ is chosen at a step 3.

**Step 3:** Consider the root locus of $1 + kL(s) = 0$ where $L(s) = \frac{\bar{b}_{\mathsf{p}}(s)\bar{b}_{\mathsf{q}}(s)}{a_{\mathsf{p}}(s)s\bar{a}_{\mathsf{q}}(s)}$. Find $k^* > 0$ such that, for each $0 < k < k^*$, all the roots of $1 + kL(s) = 0$ lie in $\mathbb{C}_{<0}$. Take $b_{\mathsf{q},m-1} \leq k^*/(b_{\mathsf{p},n-1}\lambda_N)$.

Step 1 guarantees that the condition C1 of Theorem 1 is satisfied. Step 2 with any positive $b_{\mathsf{q},m-1}$ ensures that the condition C2 is fulfilled. Finally, the constant $b_{\mathsf{q},m-1}$ chosen at Step 3 guarantees that the condition C3 holds true. To see this, we first note that $L(s)$ has all zeros in $\mathbb{C}_{<0}$ and all poles of $L(s)$, except the one at the origin, belong to $\mathbb{C}_{<0}$. Following the standard root locus argument, one can see that there exists $k^* > 0$ such that, for each $0 < k < k^*$, all the roots of $1 + kL(s) = 0$ lie in $\mathbb{C}_{<0}$. If we take $b_{\mathsf{q},m-1} \leq k^*/(b_{\mathsf{p},n-1}\lambda_N)$, then it follows that all the roots of $1 + p(s)q(s)\lambda = 0$, $0 < \lambda \leq \lambda_N$ belong to $\mathbb{C}_{<0}$, which implies that the condition C3 is satisfied.

## 3. NEW AVERAGE ESTIMATOR FOR TIME-VARYING INPUTS

In this section, we consider the case where the input vector $u(t)$ is time-varying. In particular, we assume that $u^i(t)$ applied to agent $i$ can be modeled as

$$\begin{aligned}
\dot{\chi}^i &= S\chi^i,\ \chi^i(0) = \chi_0^i, \\
u^i &= R\chi^i,
\end{aligned} \tag{5}$$

where $\chi^i \in \mathbb{R}^l$ is the state vector, $S$ and $R$ are known matrices with appropriate dimensions, and $\chi_0^i$ is the unknown initial condition for $\chi^i$. Note that the model (5) covers various types of signals such as constant, polynomial, sinusoid, exponential, etc., and the initial condition $\chi_0^i$ determines the amplitude, slope, phase, etc. Let $u^i(s)$ be the Laplace transform of $u^i(t)$ given by

$$u^i(s) = \frac{b_{\mathsf{u}}^i(s)}{a_{\mathsf{u}}(s)} = R(sI_l - S)^{-1}\chi_0^i. \tag{6}$$

The proposed average estimator is given by

$$\begin{aligned}
\dot{\xi}^i &= A_\xi \xi^i + B_\xi \left(u^i - \omega^i\right), \\
\dot{\eta}^i &= A_\eta \eta^i + B_\eta \hat{u}_{\mathsf{av}}^i, \\
\omega^i &= \sum_{j\in\mathcal{N}_i} a_{ij}(v^i - v^j),\ \ v_i = K_\xi \xi^i + K_\eta \eta^i, \\
\hat{u}_{\mathsf{av}}^i &= C_\xi \xi^i,
\end{aligned} \tag{7}$$

where $\xi^i \in \mathbb{R}^n$, $\eta^i \in \mathbb{R}^m$ are the filter states, and $A_\xi$, $B_\xi$, $C_\xi$, $A_\eta$, and $B_\eta$ are matrices which have the same structures as those given in (2) and the associated coefficients are design parameters to be determined. The vectors $K_\xi$ and $K_\eta$ are to be determined as well. The structure of the proposed estimator is shown Fig. 4.

For simplicity, define $\psi^i = [\xi^i;\eta^i]$, and rewrite the average estimator (7) as

$$\begin{aligned}
\dot{\psi}^i &= A_\psi \psi^i + B_\psi(u^i - \omega^i), \\
\omega^i &= \sum_{j\in\mathcal{N}_i} a_{ij}(K_\psi \psi^i - K_\psi \psi^j), \\
\hat{u}_{\mathsf{av}}^i &= C_\psi \psi^i,
\end{aligned} \tag{8}$$

where

$$A_\psi = \begin{bmatrix} A_\xi & 0_{n\times m} \\ B_\eta C_\xi & A_\eta \end{bmatrix},\quad B_\psi = \begin{bmatrix} B_\xi \\ 0_m \end{bmatrix}$$

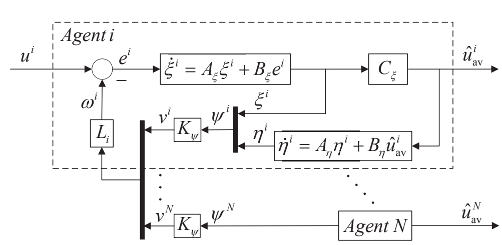


Fig. 4. Proposed dynamic average estimator for time-varying inputs.

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$$C_\psi = \begin{bmatrix} C_\xi & 0_m^\top \end{bmatrix}, \;\; K_\psi = \begin{bmatrix} K_\xi & K_\eta \end{bmatrix}.$$

Basically, the proposed estimator shares the same idea as the one proposed in Section 2, but the signals to be exchanged among agents are different; for the constant input case, agents exchange the output signal $v^i$ only (see Fig. 2), while for the time-varying case, agents exchange the scalar signal $K_\psi \psi^i$ which is a linear function of the internal states of estimator. It is emphasized that the proposed structure enables us to provide a constructive design procedure, which is the main advantage of the proposed solution over the previous result given in [17].

**Theorem 2:** Let $u(t) \in \mathbb{R}^N$ be the input vector applied to a group of $N$ agents whose $i$th component $u^i(t)$ is modeled as (5), and consider the distributed estimator (7) with $n \geq l$ and $m \geq l$. Let $a_{\mathsf{p}}(s) = \det(sI_n - A_\xi)$, $a_{\mathsf{q}}(s) = \det(sI_m - A_\eta)$, and $a_{\mathsf{u}}(s) = \det(sI_l - S)$. Then, the estimate $\hat{u}_{\mathsf{av}}(t)$ from the estimator (7) ensures that

$$\lim_{t\to\infty} \left( \hat{u}_{\mathsf{av}}(t) - \frac{1}{N} 1_N 1_N^\top u(t) \right) = 0 \tag{9}$$

if all conditions C1, C2, and C3 shown below are satisfied.

C1) $a_{\mathsf{p}}(s)$ is Hurwitz, and $a_{\mathsf{u}}(s)$ divides $b_{\mathsf{p}}(s) - a_{\mathsf{p}}(s)$.

C2) $a_{\mathsf{q}}(s) = a_{\mathsf{u}}(s)\gamma(s)$ for some stable polynomial $\gamma(s)$.

C3) $A_\psi - \lambda_i(L) B_\psi K_\psi$, $i = 2, \ldots, N$, are Hurwitz.

**Proof:** Let $\lambda_1, \ldots, \lambda_N$ be the eigenvalues of $L$. As stated in the proof for Theorem 1, $\lambda_1 = 0$, and there exists an orthonormal matrix $V$ such that $V^\top L V = \Delta = \operatorname{diag}\{0, \lambda_2, \ldots, \lambda_N\}$. We decompose $V$ as $V = \begin{bmatrix} V_1 & \cdots & V_N \end{bmatrix}$ with $V_1 = \frac{1}{\sqrt{N}} 1_N$. For simplicity, let $W = \begin{bmatrix} V_2 & \cdots & V_N \end{bmatrix}$.

Now, we define $\psi = \begin{bmatrix} \psi^1; \cdots; \psi^N \end{bmatrix}$ and collect all the dynamics (8) with $i = 1, \ldots, N$ to have

$$\begin{aligned} \dot{\psi} &= \left(I_N \otimes A_\psi - L \otimes B_\psi K_\psi\right)\psi + \left(I_N \otimes B_\psi\right) u, \\ \hat{u}_{\mathsf{av}} &= (I_N \otimes C_\psi)\psi. \end{aligned} \tag{10}$$

Let $\bar{n} = n + m$ and define $\bar{\psi} = (V^\top \otimes I_{\bar{n}})\psi$. In $\bar{\psi}$ coordinates, we rewrite (10) as

$$\begin{aligned} \dot{\bar{\psi}} =& (V^\top \otimes I_{\bar{n}}) \left(I_N \otimes A_\psi - L \otimes B_\psi K_\psi\right) (V \otimes I_{\bar{n}}) \bar{\psi} \\ &+ (V^\top \otimes I_{\bar{n}}) \left(I_N \otimes B_\psi\right) u \\ =& \left(I_N \otimes A_\psi - \Delta \otimes B_\psi K_\psi\right) \bar{\psi} + \left(V^\top \otimes B_\psi\right) u \end{aligned} \tag{11}$$

from which we have

$$\bar{\psi}(s) = \operatorname{diag}\{T_{\mathsf{av}}(s), \bar{T}(s)\} \left(\bar{\psi}(0) + \left(V^\top \otimes B_\psi\right) u(s)\right),$$

where

$$\begin{aligned} T_{\mathsf{av}}(s) &= (sI_{\bar{n}} - A_\psi)^{-1} \\ \bar{T}(s) &= \operatorname{diag}\left\{(sI_{\bar{n}} - A_\psi + \lambda_2 B_\psi K_\psi)^{-1}, \right. \\ & \qquad \left. \ldots, (sI_{\bar{n}} - A_\psi + \lambda_N B_\psi K_\psi)^{-1}\right\}. \end{aligned}$$

Then, we obtain

$$\hat{u}_{\mathsf{av}}(s) = \hat{u}_{\mathsf{av},0}(s) + \hat{u}_{\mathsf{av},u}(s), \tag{12}$$

where

$$\begin{aligned} \hat{u}_{\mathsf{av},0}(s) &= (V \otimes C_\psi) \operatorname{diag}\{T_{\mathsf{av}}(s), \bar{T}(s)\} \bar{\psi}(0) \\ \hat{u}_{\mathsf{av},u}(s) &= (V \otimes C_\psi) \operatorname{diag}\{T_{\mathsf{av}}(s), \bar{T}(s)\} \left(V^\top \otimes B_\psi\right) u(s). \end{aligned}$$

In order to find out the steady state behavior of $\hat{u}_{\mathsf{av}}(t)$, we investigate the signals $\hat{u}_{\mathsf{av},0}(t)$ and $\hat{u}_{\mathsf{av},u}(t)$, separately.

Consider the signal $\hat{u}_{\mathsf{av},0}(t)$ and compute

$$\begin{aligned} \hat{u}_{\mathsf{av},0}(t) =& \mathcal{L}^{-1}\left\{(V \otimes C_\psi) \operatorname{diag}\{T_{\mathsf{av}}(s), \bar{T}(s)\} \bar{\psi}(0)\right\} \\ =& \mathcal{L}^{-1}\left\{\left(\frac{1}{\sqrt{N}} 1_N \otimes C_\psi\right) \begin{bmatrix} T_{\mathsf{av}}(s) & 0 \end{bmatrix} \bar{\psi}(0)\right\} \\ &+ \mathcal{L}^{-1}\left\{(W \otimes C_\psi) \begin{bmatrix} 0 & \bar{T}(s) \end{bmatrix} \bar{\psi}(0)\right\}. \end{aligned}$$

Noting that

$$\begin{aligned} &\left(\frac{1}{\sqrt{N}} 1_N \otimes C_\psi\right) \begin{bmatrix} T_{\mathsf{av}}(s) & 0 \end{bmatrix} \bar{\psi}(0) \\ &= \left(\frac{1}{N} 1_N 1_N^\top \otimes C_\xi (sI_n - A_\xi)^{-1}\right) \begin{bmatrix} \xi^1(0); \ldots; \xi^N(0) \end{bmatrix}, \end{aligned}$$

it follows from the stability of $A_\xi$ (condition C1) and $A_\psi - \lambda_i B_\psi K_\psi$, $i = 2, \ldots, N$, (condition C3) that $\lim_{t\to\infty} \hat{u}_{\mathsf{av},0}(t) = 0$.

Since $\hat{u}_{\mathsf{av},0}(t)$ decays to zero, the steady state behavior of $\hat{u}_{\mathsf{av}}(t)$ is determined by that of $\hat{u}_{\mathsf{av},u}(t)$. Towards this, we decompose $\hat{u}_{\mathsf{av},u}(s)$ as follows

$$\begin{aligned} \hat{u}_{\mathsf{av},u}(s) &= (V \otimes C_\psi) \operatorname{diag}\{T_{\mathsf{av}}(s), \bar{T}(s)\} (V^\top \otimes B_\psi) u(s) \\ &= \sum_{i=1}^N V_i V_i^\top T_{\psi,i}(s) u(s), \end{aligned} \tag{13}$$

where $T_{\psi,i}(s) = C_\psi (sI_{\bar{n}} - A_\psi + \lambda_i B_\psi K_\psi)^{-1} B_\psi$.

We claim that for $i = 1, \ldots, N$, it holds that

$$T_{\psi,i}(s) = \frac{a_{\mathsf{q}}(s) b_{\mathsf{p},i}(s)}{a_{\psi,i}(s)}, \tag{14}$$

where

$$\begin{aligned} a_{\psi,i}(s) &= \det(sI_{\bar{n}} - A_\psi + \lambda_i B_\psi K_\psi) \\ b_{\mathsf{p},i}(s) &= \det \begin{bmatrix} sI_n - A_\xi + \lambda_i B_\xi K_\xi & -B_\xi \\ C_\xi & 0 \end{bmatrix}. \end{aligned}$$

To prove the claim, it is sufficient to prove that the numerator of $T_{\psi,i}(s)$ can be factored into $a_{\mathsf{q}}(s) b_{\mathsf{p},i}(s)$. In fact, one can directly obtain this result from the identity

$$\begin{aligned} &\det \begin{bmatrix} sI_n - A_\xi + \lambda_i B_\xi K_\xi & \lambda_i B_\xi K_\eta & -B_\xi \\ -B_\eta C_\xi & sI_m - A_\eta & 0_m \\ C_\xi & 0_m^\top & 0 \end{bmatrix} \\ &= \det \begin{bmatrix} sI_n - A_\xi + \lambda_i B_\xi K_\xi & -B_\xi \\ C_\xi & 0 \end{bmatrix} \det(sI_m - A_\eta). \end{aligned}$$

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From (13), (14), and (6), one has

$$\hat{u}_{\mathsf{av},u}(s) = \sum_{i=1}^{N}\left(V_iV_i^\top \frac{a_{\mathsf{q}}(s)b_{\mathsf{p},i}(s)}{a_{\psi,i}(s)}\right)\frac{1}{a_{\mathsf{u}}(s)}b_{\mathsf{u}}(s), \quad (15)$$

where $b_{\mathsf{u}}(s) = [b_{\mathsf{u}}^1(s);\cdots;b_{\mathsf{u}}^N(s)]$. Since $a_{\mathsf{u}}(s)$ divides $a_{\mathsf{q}}(s)$ (condition C2), there exists a polynomial $\gamma(s)$ such that $a_{\mathsf{q}}(s) = a_{\mathsf{u}}(s)\gamma(s)$, which yields

$$\hat{u}_{\mathsf{av},u}(s) = \sum_{i=1}^{N}\left(V_iV_i^\top \frac{\gamma(s)b_{\mathsf{p},i}(s)}{a_{\psi,i}(s)}\right)b_{\mathsf{u}}(s).$$

Moreover, by the condition C3, it holds that $a_{\psi,i}(s)$ is Hurwitz for all $i = 2,\dots,N$, which results in that

$$\mathcal{L}^{-1}\left\{\sum_{i=2}^{N} V_iV_i^\top T_{\psi,i}(s)u(s)\right\} \to 0 \text{ as } t \to 0. \quad (16)$$

Meanwhile, from the condition C1, there exists a polynomial $\bar{\gamma}(s)$ such that $b_{\mathsf{p}}(s) = a_{\mathsf{p}}(s)+\bar{\gamma}(s)a_{\mathsf{u}}(s)$. Noting that $b_{\mathsf{p},1}(s) = b_{\mathsf{p}}(s)$ and $a_{\psi,1}(s) = a_{\mathsf{p}}(s)a_{\mathsf{q}}(s)$, we have, for the term with $i = 1$ in (13),

$$\begin{aligned} T_{\psi,1}(s)u(s) &= \frac{a_{\mathsf{q}}(s)b_{\mathsf{p}}(s)}{a_{\mathsf{p}}(s)a_{\mathsf{q}}(s)}\frac{1}{a_{\mathsf{u}}(s)}b_{\mathsf{u}}(s) \\ &= \frac{a_{\mathsf{p}}(s)+\bar{\gamma}(s)a_{\mathsf{u}}(s)}{a_{\mathsf{p}}(s)}\frac{1}{a_{\mathsf{u}}(s)}b_{\mathsf{u}}(s) \\ &= u(s)+\frac{\bar{\gamma}(s)}{a_{\mathsf{p}}(s)}b_{\mathsf{u}}(s). \end{aligned}$$

Applying this result and the properties $\lim_{t\to}\hat{u}_{\mathsf{av},0}(t) = 0$ and (16), one has

$$\begin{aligned} &\lim_{t\to\infty}(\hat{u}_{\mathsf{av}}(t) - 1_N u_{\mathsf{av}}(t)) \\ &= \lim_{t\to\infty}\mathcal{L}^{-1}\left\{\left(V_1V_1^\top T_{\psi,1}(s)\right)u(s) - \frac{1}{N}1_N1_N^\top u(s)\right\} \\ &= \lim_{t\to\infty}\mathcal{L}^{-1}\left\{\frac{1}{N}1_N1_N^\top\frac{\bar{\gamma}(s)}{a_{\mathsf{p}}(s)}b_{\mathsf{u}}(s)\right\} \\ &= 0 \end{aligned}$$

which completes the proof. □

**Remark 2:** The proposed average estimator has $n+m$ states and if we have $N$ inputs, there are $N(n+m)$ states in the closed-loop system. We note that the closed-loop system (10) has $N(n+m)-l$ stable eigenvalues (by conditions C1 and C3) and the rest $l$ eigenvalues are those of $S$. This means that the signals in the closed-loop system can oscillate or even diverge depending on the model of inputs.

Now, we provide a constructive design procedure guaranteeing three conditions of Theorem 2.

**Design Procedure for Time-Varying Inputs**

**Step 1:** From the model of input signal, choose $n$ and $m$ (the orders of $a_{\mathsf{p}}(s)$ and $a_{\mathsf{q}}(s)$, respectively) such that $n \geq l$ and $m \geq l$ ($l$ is the order of $a_{\mathsf{u}}(s)$).

**Step 2:** Choose a stable monic polynomial $a_{\mathsf{p}}(s)$ with order $n \geq l$.

**Step 3:** Choose an $(n-l)$th order polynomial $\bar{\gamma}(s)$ of the form $\bar{\gamma}(s) = -(s^{n-l}+\bar{\gamma}_{n-l-1}s^{n-l-1}+\cdots+\bar{\gamma}_0)$, and take $b_{\mathsf{p}}(s) = a_{\mathsf{p}}(s)+a_{\mathsf{u}}(s)\bar{\gamma}(s)$.

**Step 4:** Choose an $(m-l)$th order polynomial $\gamma(s)$ such that all the roots of $\gamma(s)$ are different from those of $a_{\mathsf{p}}(s)$, and take $a_{\mathsf{q}}(s) = a_{\mathsf{u}}(s)\gamma(s)$.

**Step 5:** Let $P > 0$ be a solution to the Riccati equation given by

$$A_\psi^\top P + PA_\psi + I_{\bar{n}} - PB_\psi B_\psi^\top P = 0. \quad (17)$$

Let $\delta$ be such that $\delta \leq \lambda_2(L)$. Choose the feedback gain $K_\psi = \max\{1,\delta^{-1}\}B_\psi^T P$.

**Remark 3:** We emphasize that the proposed filter explicitly employ the information of input signals $u^i$'s. In fact, the internal model of inputs, i.e., $S$, is used to construct the polynomial $a_{\mathsf{u}}(s)$ and this polynomial is used to construct the matrix $A_\eta$ (condition C2) and the dynamics of $\xi$, i.e., $A_\xi$, $B_\xi$, and $C_\xi$ (condition C1).

**Remark 4:** We note that Steps 1∼4 ensure that $(A_\psi, B_\psi)$ is stabilizable, which can be proved by employing the PBH test [25]. Indeed, first note that $(A_\xi, B_\xi)$ and $(A_\eta, B_\eta)$ are controllable and let $\lambda$ be an eigenvalue of $A_\eta$ such that $\lambda \in \mathbb{C}_{\geq 0}$ where $\mathbb{C}_{\geq 0} = \mathbb{C}\setminus\mathbb{C}_{<0}$. Then, one has

$$\begin{aligned} &\text{rank}\begin{bmatrix} A_\xi-\lambda I_n & 0_n0_m^\top & B_\xi \\ B_\eta C_\xi & A_\eta-\lambda I_m & 0_m \end{bmatrix} \\ &= \text{rank}\begin{bmatrix} A_\xi-\lambda I_n & 0_n0_m^\top & B_\xi \\ 0_n0_n^\top & A_\eta-\lambda I_m & -B_\eta C_\xi(A_\xi-\lambda I_n)^{-1}B_\xi \end{bmatrix} \\ &= n+\text{rank}\begin{bmatrix} A_\eta-\lambda I_m & B_\eta\frac{b_{\mathsf{p}}(\lambda)}{a_{\mathsf{p}}(\lambda)} \end{bmatrix} \\ &= n+m, \end{aligned}$$

where the last equality follows from the facts that $(A_\eta, B_\eta)$ is controllable and $b_{\mathsf{p}}(\lambda) = a_{\mathsf{p}}(\lambda) \neq 0$. Thus, the assertion is proved.

**Remark 5:** The stabilizability ensured by Steps 1∼4 guarantee that there exist a unique solution $P$, which is symmetric and positive definite, to the Riccati equation (17), and the condition C3 in Theorem 2 is satisfied by the gain matrix $K_\psi$ determined using the solution $P$; see [26] for details.

## 4. NUMERICAL EXAMPLE

In this section, we explain the design of estimators developed in previous sections and provide simulation results. We consider a group of four agents whose network topology is shown in Fig. 5. The eigenvalues of

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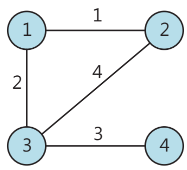


Fig. 5. Communication topology among four agents.

the Laplacian matrix $L$ are $\lambda_1 = 0$, $\lambda_2$=2.82, $\lambda_3 = 4.81$, and $\lambda_4 = 12.36$.

### 4.1. Dynamic average consensus for constant inputs

Consider the case where the inputs to the agents are constants. Let the input vector be given by $u(t) = \bar{u} = \left[10; 4; 2; 8\right]$.

According to Step 1 of the design procedure in Section 2, we choose $a_{\mathrm{p},1} = 16$, $a_{\mathrm{p},0} = 60$ to obtain $a_{\mathrm{p}}(s) = s^2 + 16s + 60$. And we take $b_{\mathrm{p}}(s) = 2s + 60$ because we choose $\bar{b}_{\mathrm{p},0} = 30$, $b_{\mathrm{p},1} = 2$. From Steps 2 and 3, we take $\bar{a}_{\mathrm{q}}(s) = s^2 + 35s + 300$, and $\bar{b}_{\mathrm{q}}(s) = s^2 + 29s + 204$. Finally, from the root locus of $1 + k\frac{(s+30)(s^2+29s+204)}{(s^2+16s+60)s(s^2+35s+300)} = 0$, we choose $k^* = 50$ and take $b_{\mathrm{q},2} = 2 < \frac{k^*}{b_{\mathrm{p},1}\lambda_4}$. Thus, the resulting transfer functions $p(s)$ and $q(s)$ are given by

$$p(s) = \frac{2s+60}{s^2+16s+60}, \quad q(s) = \frac{2(s^2+29s+204)}{s(s^2+35s+300)}.$$

The simulation result in Fig. 6 shows that the agent's estimates converge to the true average for constant inputs.

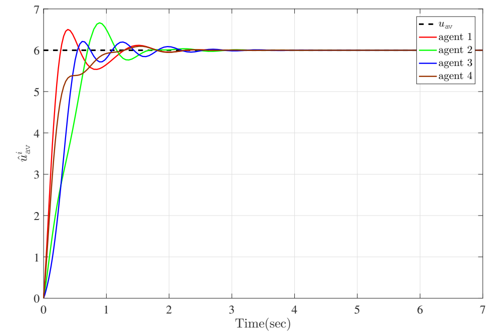


Fig. 6. Average estimate $\hat{u}^i_{\mathsf{av}}(t)$ and the average of $u(t)$ for the constant inputs.

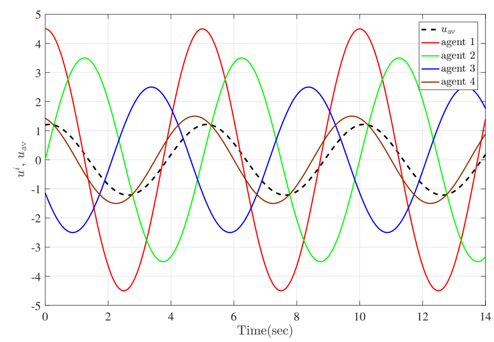


Fig. 7. The trajectories of $u^i(t)$ and their average $u_{\mathsf{av}}$ for the sinusoidal inputs.

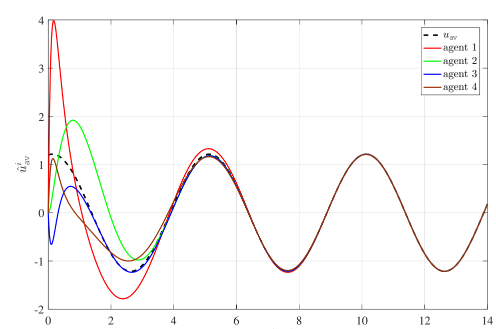


Fig. 8. Average estimate $\hat{u}^i_{\mathsf{av}}(t)$ and the average of $u(t)$ for the sinusoidal inputs.

### 4.2. Dynamic average consensus for time-varying inputs

Consider the case where the inputs of four agents are all sinusoids expressed by $u^i(t) = M^i \sin(\omega t + \phi^i)$, $i = 1, \ldots, 4$, where $\omega = \frac{2\pi}{5}$, $M = \left[M^1; \cdots; M^4\right] = \left[4.5; 3.5; 2.5; 1.5\right]$, and $\phi = \left[\phi^1; \cdots; \phi^4\right] = [0.5\pi;\ 0;\ 1.15\pi;\ 0.6\pi]$. The trajectories of $u^i(t)$ and $u_{\mathsf{av}}(t)$ are shown in Fig. 7. For these inputs, we assume that we know the internal model such that $a_{\mathrm{u}}(s) = s^2 + \omega^2$.

We follow the design procedure in Section 3. At Step 1, we take $n = 2$, $m = 3$. At Step 2, we choose $a_{\mathrm{p}}(s) = s^2 + 15s + 50$. At Step 3, we take $\bar{\gamma}(s) = -1$ to have $b_{\mathrm{p}}(s) = a_{\mathrm{p}}(s) - a_{\mathrm{u}}(s) = 15s + (50 - \omega^2)$. At Step 4, we take $\gamma(s) = s + 1$ so that $a_{\mathrm{q}}(s) = s^3 + s^2 + \omega^2 s + \omega^2$. Finally, we solve the equation (17) for $P$ and take $K_\psi = \delta^{-1} B_\psi^\top P = \left[0.28 \quad 0.42 \quad 0.09 \quad 0.45 \quad 0.4\right]$ by Step 5.

The simulation result in Fig. 8 shows that the proposed estimator successfully achieves dynamic average consensus for sinusoidal inputs.

## 5. CONCLUSION

By reinterpreting the PI-type average estimator and its generalizations, new conditions of dynamic average consensus are derived and a simpler structure for average estimator is proposed. The proposed structure admits a constructive design procedure and it requires the range of eigenvalues of the associated Laplacian matrix. Future research topics include dynamic average consensus on discrete-time signals and application to distributed Kalman filtering problem, and results on these topics will be reported in forthcoming publications.

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**Juwon Lee** received his B.S. degree in School of Robotics from Kwangwoon University, Seoul, Korea in 2015. He is currently working toward a Ph.D. degree at Kwangwoon University. His research interests include multi agent systems.

**Juhoon Back** received his B.S. and M.S. degrees in Mechanical Design and Production Engineering from Seoul National University, in 1997 and 1999, respectively. He received his Ph.D. degree from the School of Electrical Engineering and Computer Science, Seoul National University in 2004. Since 2008 he has been with Kwangwoon University, Korea, where he is currently a professor. His research interests include control system theory and design, renewable energy systems, and multi agent systems.

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